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MOTION OF TRAINS OF VORTEX RINGS

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A numerical experiment on investigation of the dynamics of the interaction of three coaxial vortex rings formed from thermics is performed. As a mathematical model of the phenomenon, the system of Navier–Stokes equations for a compressible heat-conducting gas in a cylindrical coordinate system is chosen. The calculations are performed using a finite-difference method based on an explicit three-step scheme of splitting by physical processes. The influence of the position of the thermics on the common axis of symmetry on the character of their motion and interaction is investigated.

In recent times, much attention has been focused on the problems of the gas hydrodynamics of ordered vortex flows, which is explained by the fact that these flows play an important part in many technological processes used in different industries; in particular, ring vortices can be used in practice to remove deleterious impurities released in the course of production processes [1, 2]. As laboratory, full-scale, and numerical experiments have shown, an even more efficient means of canalization of ejections of harmful output is a chain of ring vortices that, in the case of favorable relations between their dimensions and intensities and the repetition frequency of the individual vortices, can form a vortex pole that is continuous in space and performs accelerated removal of the impurity to the upper troposphere (see, for example, [3, 4]). This vortex pole is formed in the case of realization of so-called leapfrog of vortices, in which backward vortex rings alternately overtake forward ones. Such a pole is very unstable in the sense that when the conditions of realization of a leapfrog are violated, for example, in the case where the distance between any neighboring rings is increased disproportionately, it breaks up into individual trains of vortex rings. Such trains of vortex rings will be considered here.

It was established earlier that for a pair of coaxial vortex rings ($N = 2$), two types of interaction can be realized. The first type is leapfrog of vortices, in which the backward ring (which is decreased in transverse dimensions due to mutual induction) catches up with the forward ring (which is increased in transverse dimensions for the same reason) and passes through its interior transit, thus overtaking the latter; in the ideal-liquid approximation, the process of overtaking of the forward ring by the backward ring is infinite and periodic [5, 6]; however, in a viscous medium, this process is finite due to diffusion of the vorticity [7, 8]. The second type is a single interaction of the vortex rings, as a result of which confluence of them occurs with formation of a monovortex (see [5–8]).

1. In the motion of chains of coaxial vortex rings ($N > 2$), the interaction between individual objects is more complex in character than in the case of a tandem of rings, including in the ideal-liquid approximation. This is due to the fact that such a system of rings is very sensitive to the flow-field randomness established with time, which is explained by the strong nonlinearity of the initial Euler equations and the dynamic Hamilton system following from them.

In [6], how the concept of chaos manifests itself in the case of interaction of three thin vortex rings ($N = 3$) was investigated. Within the limits of the model of an ideal liquid, such a system is classified among conservative physical systems, in which all dynamic systems of classical mechanics are included. A special

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feature of such systems is conservation of their phase volume, which differentiates them from the dissipative analogs that will be investigated below.

It is easiest to reveal the existence of chaos in a system of vortex rings by considering their trajectories: chaos occurs when the trajectories are irregular or nonperiodic in character. Moreover, there are special methods that make it possible to verify or refine the character of the motion. One of them is the Poincaré method of mapping [9] (such a mapping is obtained when the trajectories in a d-dimensional phase space intersect a (d–1)-dimensional hyperplane; in this case, chaos manifests itself as a disordered pattern of intersection points). Another method is the Lyapunov method, using which we can determine the index λ (the Lyapunov index), which is an important quantitative characteristic of chaotic motion – a measure of the exponential scattering of the initially closely spaced trajectories (for $\lambda = 0$ the motion is ordered, and for $\lambda > 0$ the motion is chaotic). For example, within the limits of the model of an ideal liquid, the motion of a system of two coaxial vortex rings is always ordered: the maximum Lyapunov index for a tandem of rings approaches zero, while the Poincaré cross section consists of several points; the number of parameters determining the motion of the system is equal to three: R_2/R_1, h, and Γ_2/Γ_1 (here R_1 and R_2 are the radii of the lower and upper rings, respectively, Γ_1 and Γ_2 are the intensities of their vorticity, and h is the distance between them at $t = 0$). However, for a system of three coaxial rings, the number of determining parameters increases to six, which makes the classification of the interaction very difficult. Because of this, in investigating the interaction of three vortex rings it makes sense to restrict the consideration to comparatively simple examples, on which we can show, in particular, in what situations the motion can be ordered.

In [6], a number of variants of the motion of a system of three thin rings (the ratio of the radius of the cross section of a ring to its proper radius is one to one hundred) are considered. Certain cases in which an ordered regime of motion of the system is established are revealed. For example, for three rings equal in dimensions and intensity and equally spaced at $t = 0$, a regime where initially the lower ring lags behind the middle and upper rings and for the latter rings the leapfrog regime appears is established after chaos that is finite in time. However, if one determining parameter of the problem, such as the initial distance between any pair of rings in the system, is changed slightly, the motion of the system becomes chaotic at any t (see [6] for details). The main result of the numerical experiment performed in [6] is the conclusion that in the case of the existence of chaos, it is impossible to predict the character of the motion of a system of three vortex rings even for cases with close initial conditions, while for a system of two vortex rings, such a prediction can easily be made.

In [10], an example of using the Poincare method of mapping to investigate the motion of a system of three thin vortex rings that are equal in dimensions and are equally spaced along the height, in which the two lower rings have the same intensities of vorticity ($\Gamma_2 = \Gamma_1 = \Gamma$) and the intensity of vorticity of the upper ring is equal to zero ($\Gamma_3 = 0$), is presented with a rigorous proof of the randomness of the behavior of such a system.

In the present work, we investigate numerically the motion of a train of three coaxial thermics that are motionless at the initial instant in the approximation of a viscous compressible fluid.

2. Assume that at the initial instant of time $t = 0$, there are three thermics of the same radius $R_1^* = R_2^* = R_3^* = 1350$ m with centers on one vertical axis (the z axis), and the temperature is distributed by the law

$$T = T_0 + (T_1^* - T_0) \exp\left[-\left(4\frac{R}{R_1^*}\right)^2\right]. \tag{1}$$

Here T_1^* is the maximum temperature of the gas in a thermic (which is the same in all three thermics), T_0 is the temperature at the surface of the earth, and R is the distance from a point inside a thermic to its center, which is determined from the formula $R = \sqrt{(x - x_i)^2 + (y - y_i)^2}$, $i = 1, 2, 3$.

As a mathematical model, we choose the system of Navier–Stokes equations for a compressible heat-conducting gas in a cylindrical coordinate system for the axisymmetric case (r, z) with the equation of state for a perfect gas (here the continuity equation is replaced by the equation for pressure)

$$\frac{d\mathbf{V}}{dt} = -\frac{1}{\rho}\nabla p + \mathbf{G} + \frac{1}{\rho}\left[\nabla(\mu(\nabla\cdot\mathbf{V})) + \frac{1}{3}\nabla(\mu \operatorname{div}\mathbf{V})\right],$$

$$\frac{dT}{dt} = -\frac{AT}{c_v}\operatorname{div}\mathbf{V} + \frac{AT}{c_v}\operatorname{div}(k\nabla T), \tag{2}$$

$$\frac{dp}{dt} = -\gamma p \operatorname{div}\mathbf{V} + \frac{A}{c_v}\operatorname{div}(k\nabla T), \quad \frac{d}{dt} = \frac{\partial}{\partial t} + (\mathbf{V}\cdot\nabla), \quad \mathbf{G} = (0, g),$$

$$p = A\rho T. \tag{3}$$

The problem is solved in the rectangular region $S(t) = \{0 \le r \le f(t),\ 0 \le z \le \varphi(t)\}$ with moving right and upper boundaries. The boundary conditions are as follows:

$$r = 0: \quad u = \frac{\partial v}{\partial r} = \frac{\partial T}{\partial r} = \frac{\partial p}{\partial r} = 0, \quad r = f(t): \quad \frac{\partial u}{\partial r} = \frac{\partial v}{\partial r} = \frac{\partial T}{\partial r} = \frac{\partial p}{\partial r} = 0,$$

$$z = 0: \quad u = v = \frac{\partial T}{\partial z} = 0, \quad z = \varphi(t): \quad \frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = \frac{\partial T}{\partial z} = \frac{\partial p}{\partial z} = 0. \tag{4}$$

The rise of the thermics is investigated within the model of a standard (table) atmosphere [8].

As initial conditions, we add the following conditions to (1): $u = v = 0$ and $p = p_a(z)$ throughout the region $S(0)$ and $T = T_a(z)$ outside the thermics (here $p_a(z)$ and $T_a(z)$ are the pressure and the temperature at a height z in the standard atmosphere).

We introduce the following characteristic scales: $D = 2R_1^*$ for the length, $\sqrt{D/g}$ for the time, $\sqrt{Dg}$ for the velocity, $T_0 = 288$ K for the temperature, and $\rho_0 = 1.23$ kg/m^3 for the density.

After nondimensionalization, the initial equations (2)–(3) and the boundary conditions (4) and initial conditions contain the following determining parameters of the problem:

$$\mathrm{Re} = \frac{\rho_0 D\sqrt{Dg}}{\mu}, \quad \mathrm{Pr} = \frac{c_p\mu}{k}, \quad \mathrm{M} = \sqrt{\frac{Dg}{\gamma AT}}, \quad \gamma = c_p/c_v,$$

$$R_1 = R_2 = R_3 = \frac{R_1^*}{D}, \quad T_1 = T_2 = T_3 = \frac{T_1^*}{T_0}, \quad h_{12} = \frac{h_{12}^*}{D}, \quad h_{23} = \frac{h_{23}^*}{D} \tag{5}$$

(here, the subscripts 1, 2, and 3 correspond to the lower, middle, and upper thermics and h_{12}^* and h_{23}^* are the distances between the corresponding thermics).

The dimensionless system of differential equations obtained is integrated numerically using the method of splitting by physical processes. The discretization is performed using an explicit three-step scheme: in the first and second steps, we solve the convective part of the equations, and in the third step we take into account the dissipative terms. We use grids containing 36×225 to 48×331 calculational nodes.

3. The problem posed above for trains of three thermics was solved for the following determining parameters (5): $\mathrm{Re}_{ef} = 1000$, $\mathrm{Pr} = 1$, $\mathrm{M} = 0.48$, $\gamma = 1.4$, $R_1 = R_2 = R_3 = 0.5$, $T_1 = T_2 = T_3 = 12.8$ (these

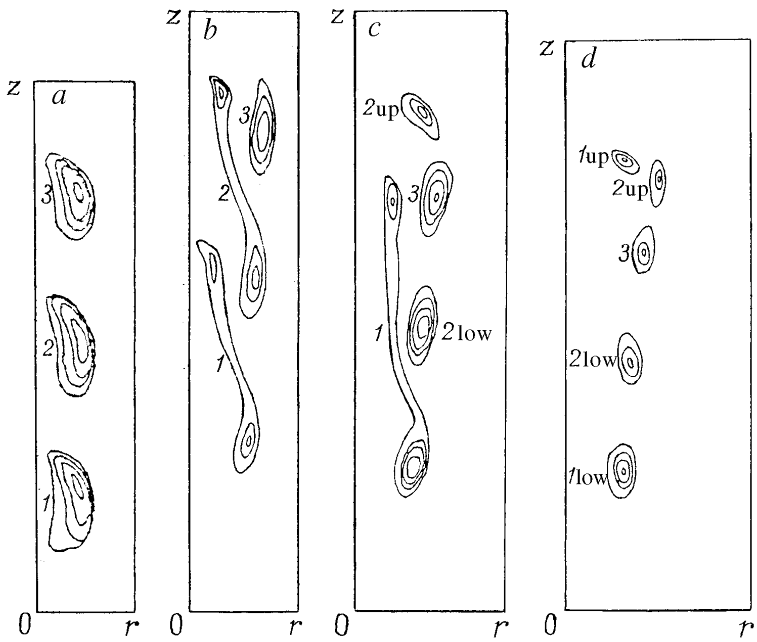

Fig. 1. Distributions of lines of equal vorticity ω for four instants of time t = 16, 50, 72, and 98 sec (a–d, respectively).

parameters were fixed). The initial distances between the thermics h_{12} and h_{23} were varied: $h_{12} = h_{23} = 1.5$; $h_{12} = 1.5$, $h_{23} = 1.75$; $h_{12} = h_{23} = 1$ (variants 1, 2, and 3, respectively).

For the problem considered, the Reynolds number calculated from the molecular viscosity (see (5)) has the order Re $\cong 10^9$–10^{10}; however, as is known, the investigated flow in thermics rapidly acquires a developed turbulent character as a result of intensification of convection (see, for example, [11]). This is taken into account by introducing, in place of the laminar coefficients of dynamic viscosity μ, a certain constant effective one μ_{ef} that models turbulence. Thus, in the system of dimensionless equations of motion, the number Re_{ef} represents a turbulent analog of the corresponding laminar criterion Re from (5) and it has a significantly lower order. The values of Re_{ef} were chosen by means of a series of calculations of the rise of a single nuclear explosion and comparison of their results with experimental data presented in [12]. It was established that $Re_{ef} \cong 10^3$. Moreover, the problem of the possible influence of the approximational viscosity of the scheme on the solution have already been considered in detail by the authors in their earlier works. Results of calculating the rise of a single thermic performed for the above-indicated parameters according to different difference schemes (with variation of the number of grid nodes) were compared, and these results were compared with the results of calculations performed according to a nonviscous model. It was established that the chosen value of the number Re_{ef} corresponds to the case where the "real" viscosity used overrides the effects introduced by the approximational viscosity.

Let us analyze the results of the calculation of variant 1. By the time t = 4.5 sec, all three thermics roll up autonomously into vortex structures (which corresponds to the characteristic time of rolling for a single thermic [11]) with practically the same distribution of the velocities and the vorticities in them (the maximum value of the vertical component of the velocity is 73 m/sec, and the maximum value of the dimensionless vorticity ($\omega = \frac{1}{2} |\mathrm{rot}\ \mathbf{V}|$) is 2.53). In this case, the minimum of the vertical component of the velocity between the thermics on the z axis is equal to 24 m/sec, i.e., the gas between the thermics is already set in motion. However, the temperatures differ significantly: $T_{1\,\max}$ = 3250 K, $T_{2\,\max}$ = 3050 K, and $T_{3\,\max}$ = 2750 K (i.e., $\Delta_{13}T$ = 500 K), which is quite natural, since the ambient air entrained in the core at the height of the upper thermic is colder than the air at the height of the lower thermic.

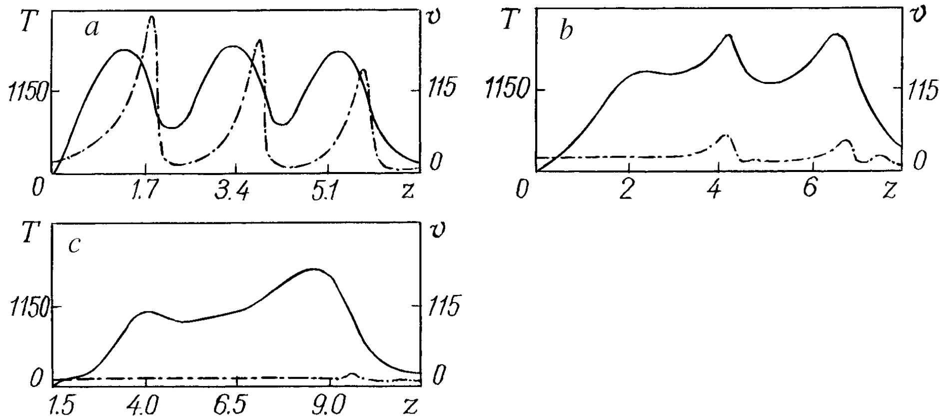

Fig. 2. Distributions of the functions for velocity $v(z)$ (solid curves) and temperature $T(z)$ (dash-dot curves) on the axis of symmetry for three instants of time t = 16, 50, and 72 sec (a–c, respectively).

Figure 1 shows the patterns of the distribution of lines of equal vorticity ω for four instants of time.

By the time t = 16 sec, there is still no interaction between the thermics, and they continue to appear as single analogs (see Fig. 1a; here, the inside isolines correspond to $\omega = 5.1$, and the outside isolines correspond to $\omega = 1.1$; $0 \le r \le 1.3$ km, $0 \le z \le 6.8$ km). The corresponding distributions of the vertical component of the velocity v and the temperature T on the axis of symmetry z are presented in Fig. 2a (here and below, the solid and dash-dot lines correspond to them). It is seen that the values of $v(z)$ ($v_{max}(z)$ = 170 m/sec) are the same in all three thermics, and the difference in temperature between them becomes even larger ($\Delta_{13}T = T_{1\,max}(z) - T_{3\,max}(z)$ = 800 K).

Then, beginning from approximately t = 30 sec, the phenomenon of a play of vortices (leapfrog) is realized. In this case, the middle ring 2 and the lower ring 1 gradually transform into vortex structures extended along the vertical with two cores in the upper and lower parts, and they are drawn into the interior transits of rings 3 and 2, respectively, driving the latter rings away from the axis of symmetry z (see Fig. 1b; t = 50 sec; $0 \le r \le 1.4$ km, $0 \le z \le 8$ km). The interaction of objects 2 and 3 occurs in the same manner as the interaction in a tandem of rings, while in the case of the interaction of objects 1 and 2, the upper core of object 1 interacts with the lower core of object 2. Here, both lower objects pass through the upper objects with the same maximum velocity $v_{max}(z)$ = 195 m/sec (see the solid curve in Fig. 2b). By this time (t = 50 sec), the thermics are already very cold: $T_{1\,max}$ = 520 K and $T_{3\,max}$ = 380 K ($\Delta_{13}T$ = 140 K).

Some time after the passage of the upper core of the middle ring 2 through the interior transit of the upper ring 3 (at t = 65 sec), it breaks with formation of two rings around its upper and lower cores (in Fig. 1c they are denoted by 2_{up} and 2_{low}, respectively, t = 72 sec; $0 \le r \le 2.2$ km, 1.5 km $\le z \le$ 11.5 km). By this time, the upper core of the lower ring 1, having passed through the interior transit of ring 2_{up}, finds itself in the transit of the former upper ring 3 (see Fig. 1c). Here, the maximum vertical component of the velocity belongs to the upper core of ring 1, and the second maximum belongs to its lower core (see the solid curve in Fig. 2c). By this time, the temperature in the thermics decreases further: $T_{1\,max}$ = 410 K and $T_{3\,max}$ = 310 K ($\Delta_{13}T$ = 100 K).

Subsequently, after the passage of the upper core of the lower thermic 1 through the interior transit of the upper ring 3, it also breaks up into two rings: the upper ring 1_{up} and the lower ring 1_{low} (at t = 80 sec).

Some time later, ring 1_{up} passes through the interior transit of ring 2_{up} and finds itself in the forefront of a system that now consists of five rings (see Fig. 1d; t = 98 sec; $0 \le r \le 3.3$ km, 1.6 km $\le z \le$ 13 km). The maximum temperatures are $T_{1\,max}$ = 360 K and $T_{3\,max}$ = 290 K ($\Delta_{13}T$ = 70 K).

During the subsequent evolution of the flow, at first (at approximately t = 125 sec), confluence of the passing rings 1_{up} and 2_{up} with the former upper ring 3 occurs, and then (at t = 150 sec) confluence of the lower rings 1_{low} and 2_{low} occurs with subsequent diffusion dispersal of both vortex structures formed.

When the initial position of the thermics is asymmetric (variant 2), at first, interaction between the two nearest rings formed from the corresponding thermics occurs. In this case, the lower ring 1 passes successively through the two upper rings 2 and 3. In the process of realization of the phenomenon of a play of vortices, at the moments of passage of the rings, a flow with high vertical velocities arises along the axis of symmetry; a vortex pole with intense vortex draft at the center is formed. Then, everything occurs qualitatively in the same manner as in variant 1.

At the initial stage of contact of the thermics (variant 3), before the beginning of interaction, they have no time to transform to vortex rings with interior transits of nonzero radius ($r > 0$); therefore, the interaction immediately represents a confluence with formation of a single vortex with triple heat energy, which rises by the law of a single thermic $z \cong t^{1/2}$.

In conclusion we note that, as the analysis of the variants presented here and other variants calculated by the authors has shown, in the compressible-gas approximation where dissipation processes are taken into account, unlike the ideal-gas approximation, there is no randomness in the motion of a system of three coaxial vortex rings, which provides support for the conclusions [6] of a stabilizing influence of viscosity.

NOTATION

N, number of vortex rings (or thermics); d, dimensionality of the phase space; R_1, R_2, and R_3, radii of the vortex rings (or thermics); $D = 2R_1$, diameter of the lower thermic; Γ_1, Γ_2, and Γ_3, intensities of the vorticity of the rings (of the circulation); t, time; $\mathbf{V}(u, v)$, velocity; ρ, density; p, pressure; T, temperature (ρ_0, p_0, and T_0 are the same on the surface of the earth); g, free-fall acceleration; μ and k, coefficients of dynamic viscosity and thermal conductivity; c_v and c_p, specific heats at constant volume and pressure; γ, adiabatic exponent; A, coefficient in the equation of state; $S(t)$, calculational region; $f(t)$ and $\varphi(t)$, right and upper boundaries of the calculational region; Re, Pr, and M, Reynolds, Prandtl, and Mach numbers; ω, vorticity.

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